Equivalence Between Tilted Formulas and Tilted Hyper-Rectangles

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Background

Symbolic Volume Integration entails decomposing a region into hyper-rectangles, integrating the hyper-rectangles individually, and summing the results. The region of integration is defined by a logical formula, usually a conjunction of inequalities; for example, the region is a convex polytope if the formula is a conjunction of linear inequalities. Thus far, the hyper-rectangles used in the algorithm have all been aligned with the axes. There is nothing wrong with this in theory, but in practice it leads to slow convergence when the boundary of the region of integration is not aligned with the axes; the hyper-rectangles can only touch the boundary with their corners, rather than with their faces.

Intuitively, this problem could be lessened if the hyper-rectangles were rotated to align with the boundary of the region—we call this a rotation of the hyper-rectangles—or if the region were rotated to align with the axes—referred to as a rotation of the formula. We will show that rotations of the formula are equivalent to rotations of the hyperrectangles.

Some Notation

In the past, we have used the following definition for the set of rectangles subsumed by formula φ.

$$\cube_\varphi \equiv \left(\bigwedge_{x \in \mathcal{X}_\varphi} l_x \le u_x \right) \wedge \forall \mathcal{X}_\varphi \cdot \left[\left(\bigwedge_{x \in \mathcal{X}_\varphi} l_x \le x \le u_x \right) \implies \varphi \right]$$

We present some new notation, and then use it to rewrite $\cube_\varphi$ in a way that lends itself to a discussion of rotations.

Whereas $\mathcal{X}_\varphi = \{x_1, \ldots, x_n\}$ denotes the set of free variables in formula φ, we introduce $\underline{\mathcal{X}}_\varphi \in \mathbb{R}^n$, the n-dimensional vector formed by the free variables of φ. Analogously, we let $\underline{l}$ and $\underline{u} \in \mathbb{R}^n$ be n-dimensional vectors.

Using this notation, we can rewrite $\Box_\varphi$ in the following form:

$$\Box_\varphi \equiv (\underline{l} \leq \underline{u}) \wedge \forall \underline{\mathcal{X}}_\varphi \cdot [(\underline{l} \leq \underline{\mathcal{X}}_\varphi \leq \underline{u}) \implies \varphi]$$

where $\leq$ is componentwise comparison.

The formula φ can also be rewritten. It is typically considered a conjunction of inequalities that define a region:

$$\varphi \equiv (\mathsf{F}_1(x_1, \ldots, x_n) \leq b_1) \wedge \cdots \wedge (\mathsf{F}_m(x_1, \ldots, x_n) \leq b_m)$$

Equivalently, we can say

$$\varphi \equiv \underline{\mathsf{F}}(\underline{\mathcal{X}}_\varphi) \leq \underline{b}$$

where $\underline{\mathsf{F}} : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function and $\underline{b} \in \mathbb{R}^m$. This may appear at first glance to be a circular definition; however, it isn't really circular since the formula φ and the vector $\underline{\mathcal{X}}_\varphi$ are both obtained directly from a fixed program.

Tilted Formulas and Tilted Hyper-Rectangles

Suppose we wish to "tilt" the formula φ, such that its corresponding region is rotated by the rotation matrix $\mathsf{R} \in \mathcal{M}_{n,n}(\mathbb{R})$. We denote the result φ_R; it is given by

$$\varphi_\mathsf{R} \equiv \underline{\mathsf{F}}(\mathsf{R}^{-1}\underline{\mathcal{X}}_\varphi) \leq \underline{b}$$

This is the case, since $\underline{\mathcal{X}}_\varphi$ satisfies φ if and only if $\mathsf{R}\underline{\mathcal{X}}_\varphi$ satisfies φ_R.

Now suppose we wish to "tilt" the set of hyperrectangles subsumed by φ, by the rotation matrix R. The result, which we call ${}_\mathsf{R}\Box_\varphi$, is given by

$${}_\mathsf{R}\Box_\varphi \equiv (\underline{l} \leq \underline{u}) \wedge \forall \underline{\mathcal{X}}_\varphi \cdot \left[\left(\underline{l} \leq \mathsf{R}^{-1}\underline{\mathcal{X}}_\varphi \leq \underline{u}\right) \implies \varphi\right]$$

This is true, since $\underline{\mathcal{X}}_\varphi$ satisfies $\Box_\varphi$ if and only if $\mathsf{R}\underline{\mathcal{X}}_\varphi$ satisfies ${}_\mathsf{R}\Box_\varphi$.

Equivalence

We now show that tilting a formula is equivalent to tilting a hyper-rectangle.

Suppose we tilt a formula φ with rotation matrix R, yielding φ_{R}. The axes-aligned hyper-rectangles subsumed by φ_{R} are given by

$$\begin{aligned}
\square_{\varphi_{\mathsf{R}}} &\equiv (\underline{\mathsf{l}} \leq \underline{\mathsf{u}}) \wedge \forall \underline{\mathcal{X}}_{\varphi} \cdot [(\underline{\mathsf{l}} \leq \underline{\mathcal{X}}_{\varphi} \leq \underline{\mathsf{u}}) \implies \varphi_{\mathsf{R}}] && (1) \\
&\equiv (\underline{\mathsf{l}} \leq \underline{\mathsf{u}}) \wedge \forall \underline{\mathcal{X}}_{\varphi} \cdot \left[(\underline{\mathsf{l}} \leq \underline{\mathcal{X}}_{\varphi} \leq \underline{\mathsf{u}}) \implies \left(\underline{\mathsf{F}}(\mathsf{R}^{-1}\underline{\mathcal{X}}_{\varphi}) \leq \underline{\mathsf{b}}\right)\right] && (2) \\
&\equiv (\underline{\mathsf{l}} \leq \underline{\mathsf{u}}) \wedge \forall \underline{\mathcal{X}}_{\varphi} \cdot \left[\left(\underline{\mathsf{l}} \leq \mathsf{R}\mathsf{R}^{-1}\underline{\mathcal{X}}_{\varphi} \leq \underline{\mathsf{u}}\right) \implies \left(\underline{\mathsf{F}}(\mathsf{R}^{-1}\underline{\mathcal{X}}_{\varphi}) \leq \underline{\mathsf{b}}\right)\right] && (3) \\
&\equiv (\underline{\mathsf{l}} \leq \underline{\mathsf{u}}) \wedge \forall \underline{\mathcal{X}}_{\varphi} \cdot [(\underline{\mathsf{l}} \leq \mathsf{R}\underline{\mathcal{X}}_{\varphi} \leq \underline{\mathsf{u}}) \implies (\underline{\mathsf{F}}(\underline{\mathcal{X}}_{\varphi}) \leq \underline{\mathsf{b}})] && (4) \\
&\equiv (\underline{\mathsf{l}} \leq \underline{\mathsf{u}}) \wedge \forall \underline{\mathcal{X}}_{\varphi} \cdot [(\underline{\mathsf{l}} \leq \mathsf{R}\underline{\mathcal{X}}_{\varphi} \leq \underline{\mathsf{u}}) \implies \varphi] && (5) \\
&\equiv_{\mathsf{R}^{-1}} \square_{\varphi} && (6)
\end{aligned}$$

In English, rotating the formula φ with matrix R is equivalent to rotating the hyper-rectangles by R^{-1} $\blacksquare$.