Weighted Model Integration
with Orthogonal Transformations

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Background: Weighted Model Integration

[Belle et al 2015, 2016; Chistikov et al 2015, 2014]

- **Inference method** for probabilistic models;
  - e.g., Bayesian networks.
  - More generally: **probabilistic programs**.
  - Restrict ourselves to linear real arithmetic.

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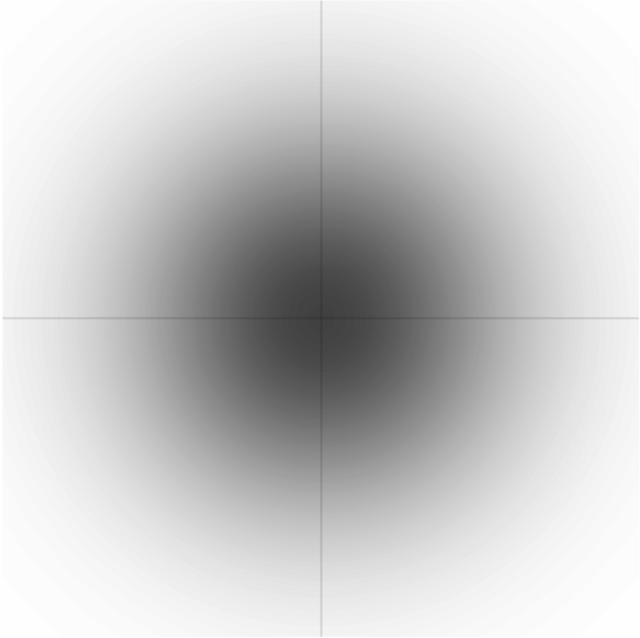

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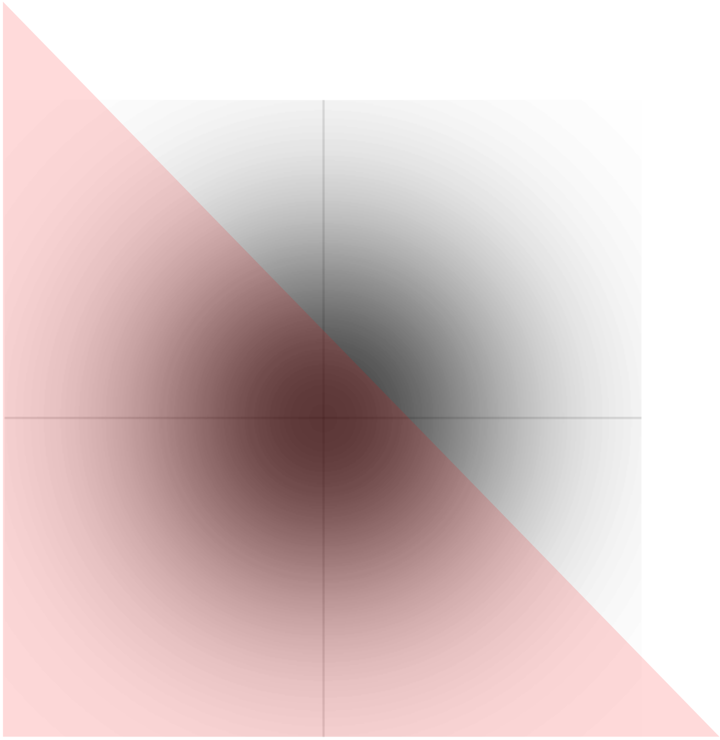

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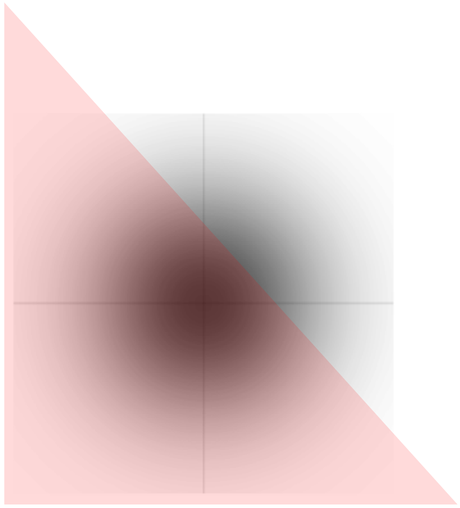

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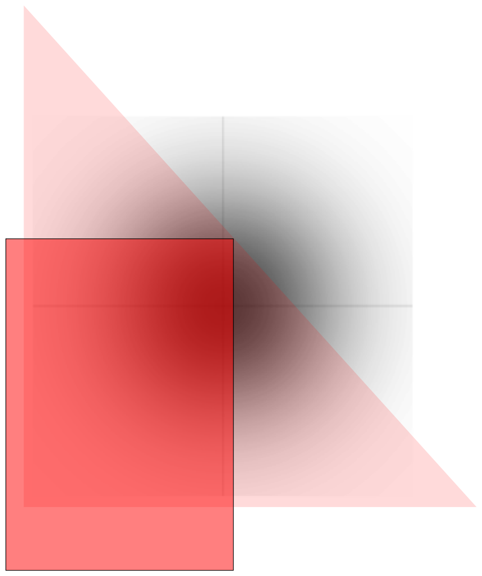

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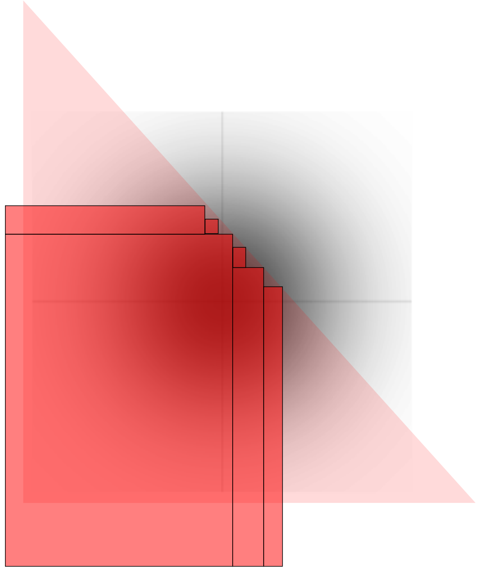

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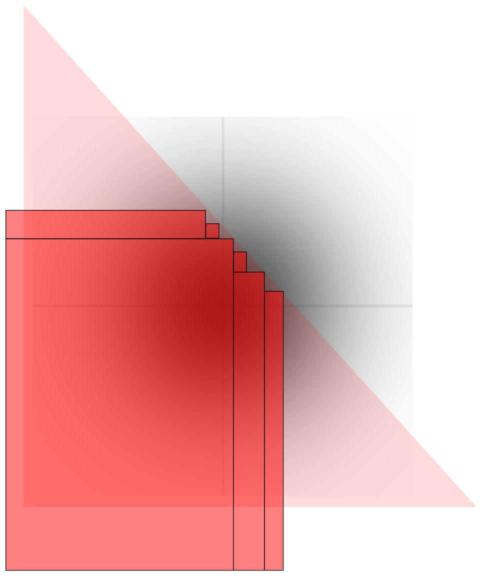

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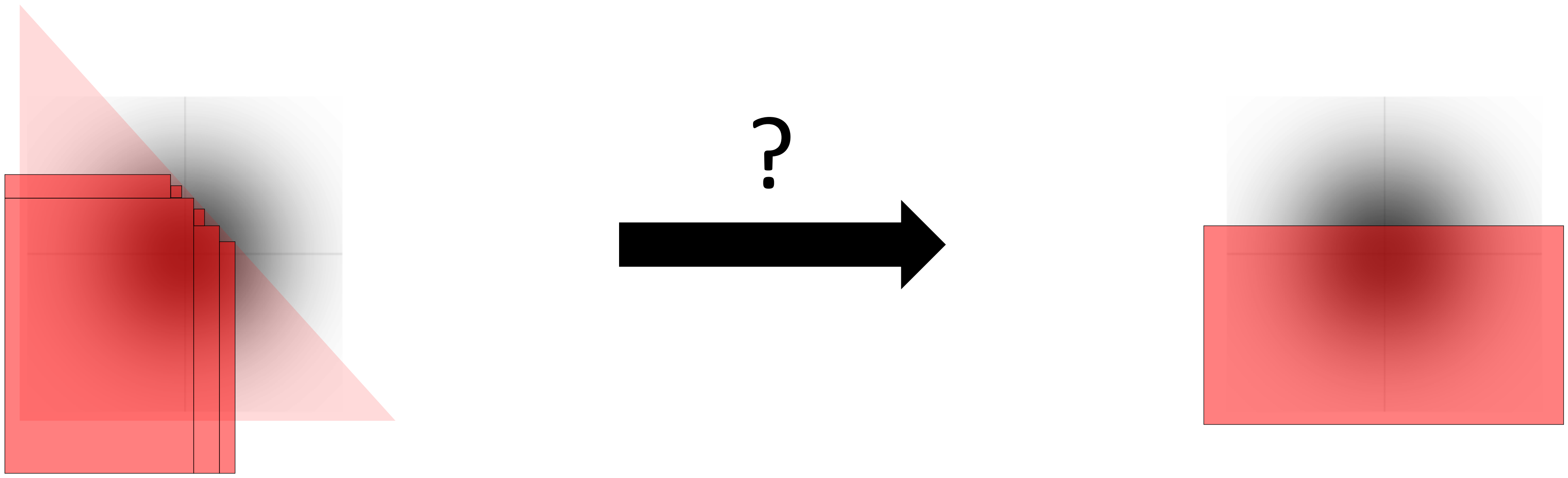
- Problem: **inefficient sampling.**
- Can we improve efficiency by **rotating** the formula**?**



Using Rotation Matrices

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- Integration by Substitution of Variables:

$$\int_{T\varphi} f(\mathcal{X})d\mathcal{X} = \int_{\varphi} f(T\mathcal{X})|\det T|d\mathcal{X}$$

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Region of integration transformed by matrix T

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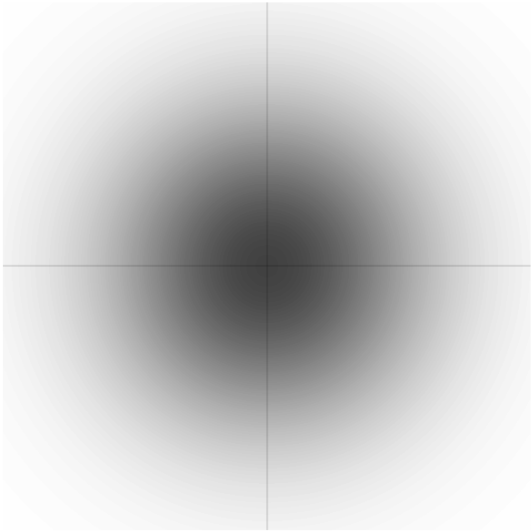
Region of integration transformed by matrix T

Determinant is 1
for orthogonal transformations.

Using Rotation Matrices

- When does a transformation **preserve stochastic independence?**
  (Rectangular decomposition requires independent random variables.)

Standard normal joint PDF



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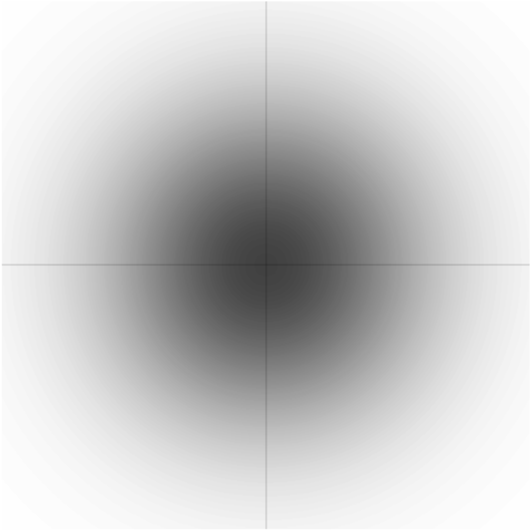

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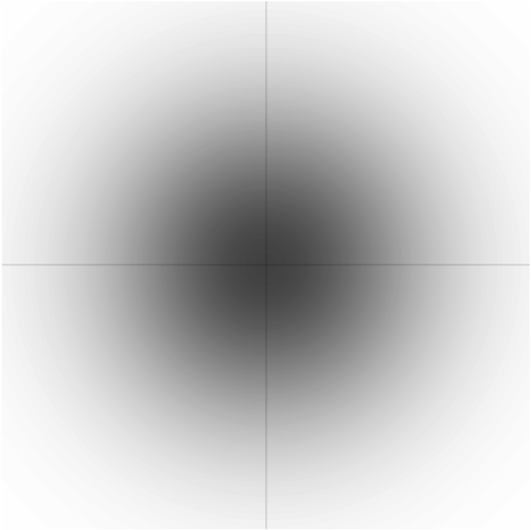

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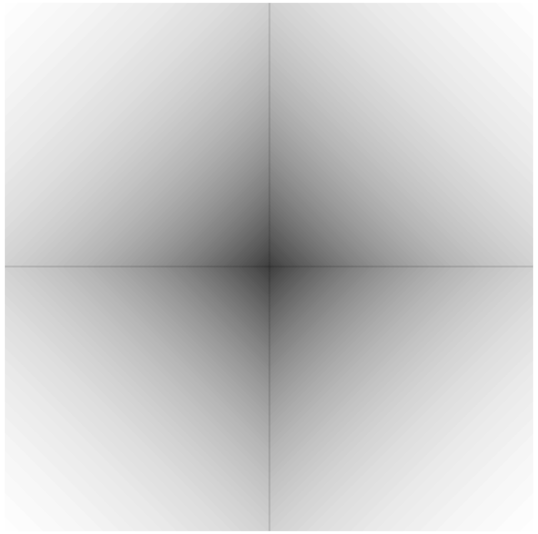

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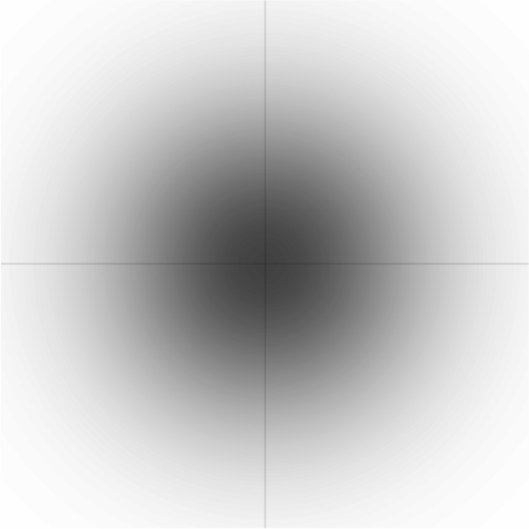

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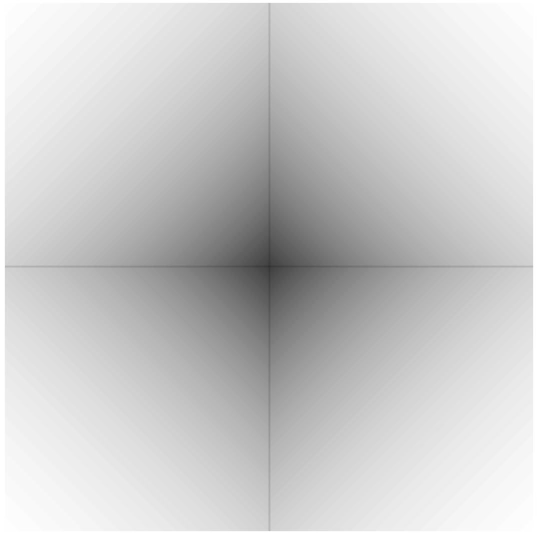
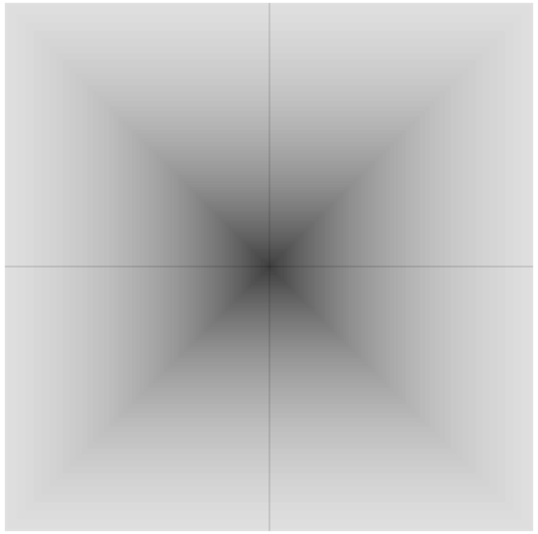


NOT a product distribution.

Rotations do **NOT** preserve independence.



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Contrapositive

$$\left.\begin{aligned} L_1 &= \sum_{i=1}^{k} \alpha_i X_i \\ L_2 &= \sum_{i=1}^{k} \beta_i X_i \end{aligned}\right\} \not\text{independent} \quad \Longleftarrow \quad X_i \not\sim N(\mu_i, \sigma_i^2)$$

Bottom line: Rectangular Decomposition can only use linear transformations on **normally distributed variables**.

Using Rotation Matrices

- A little bit of wiggle room:
  - We can restrict rotations to the “Gaussian subspace”
  - We can still handle *any* multivariate normal distribution
    - (shift and scale to standard form)

Using Rotation Matrices

- How do we make sure the transformed formula has **rational coefficients?**
- Rational **Givens Rotations**:

$$G(i,j) = \frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & a & \cdots & -b & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & b & \cdots & a & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

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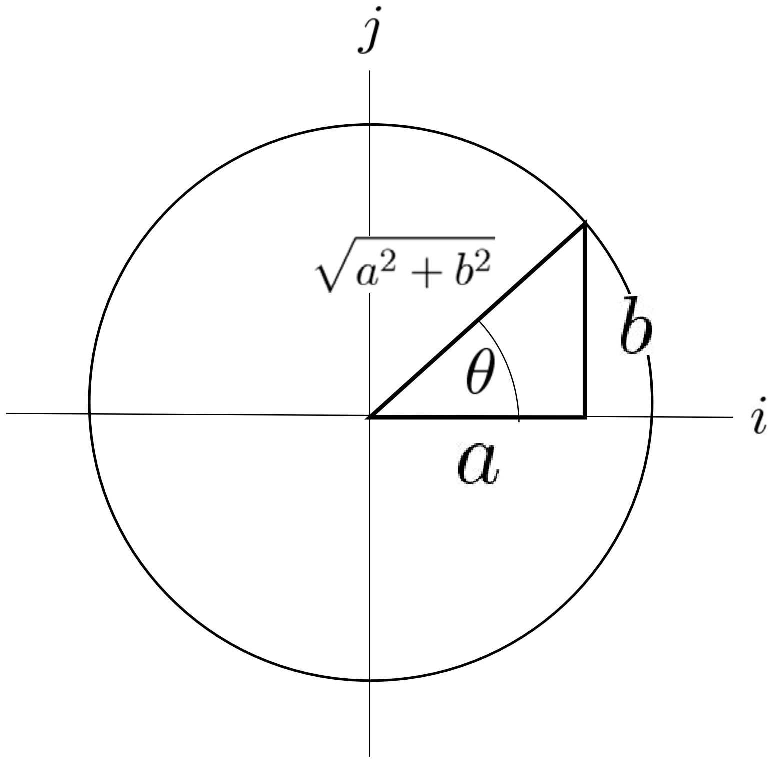

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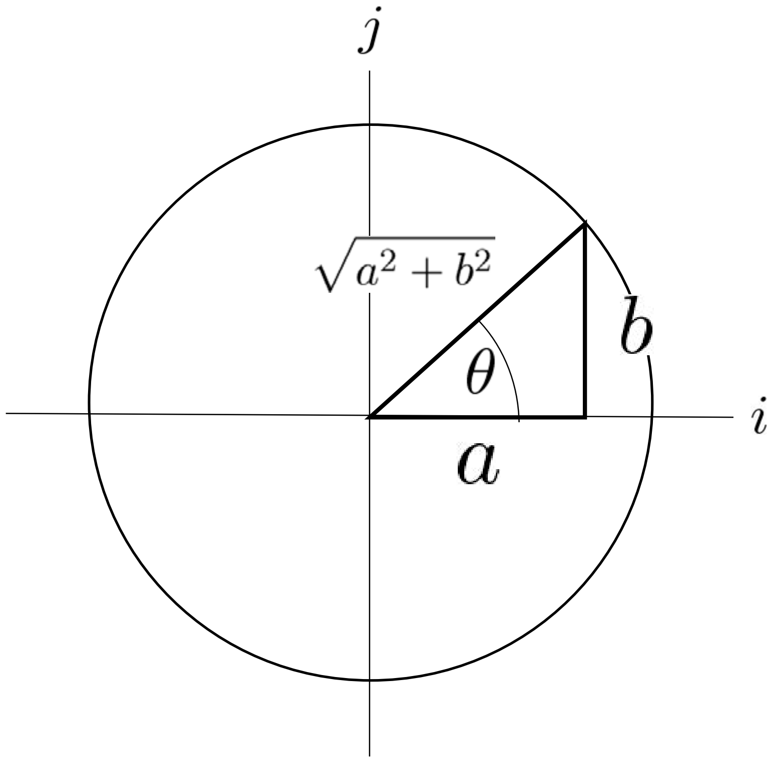

Pythagorean Triples

$$a, b, \sqrt{a^2+b^2} \in \mathbb{Z}$$

$$\Downarrow$$

$$G(i,j) \in \mathbb{Q}^{k \times k}$$

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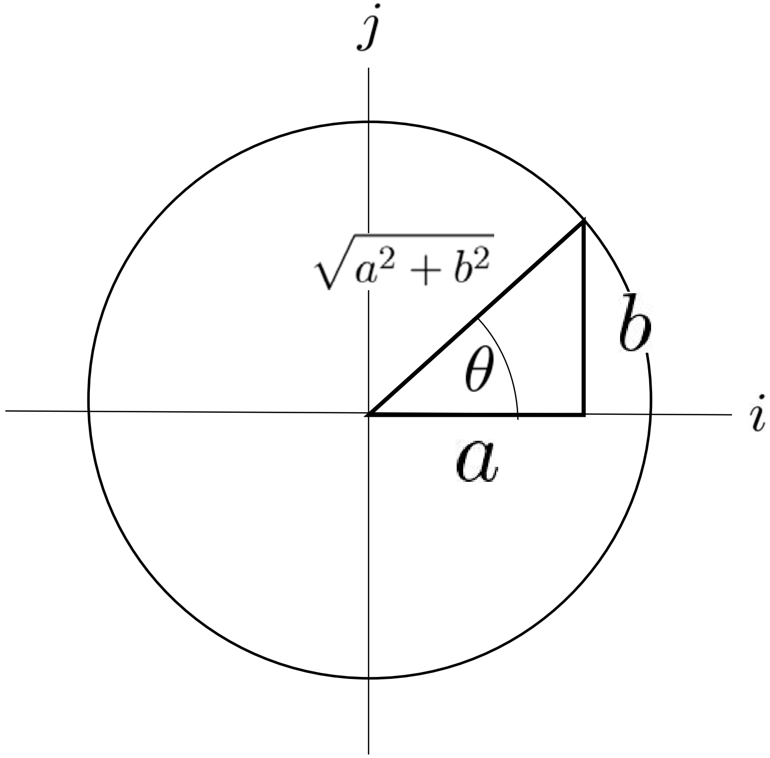
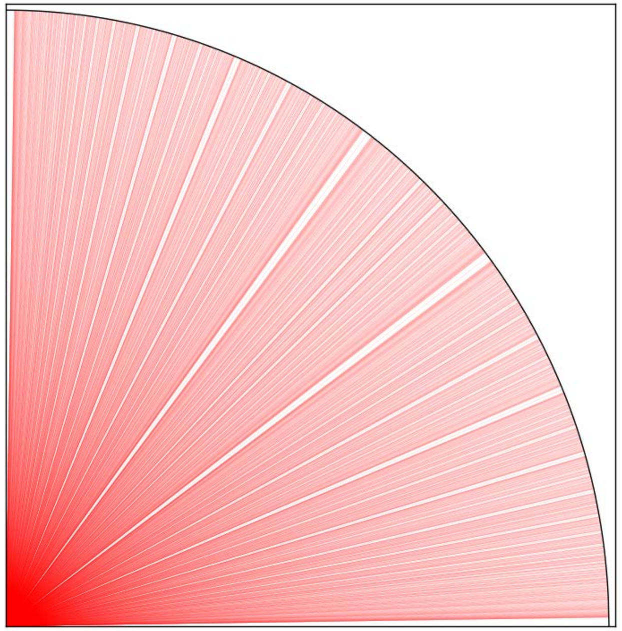
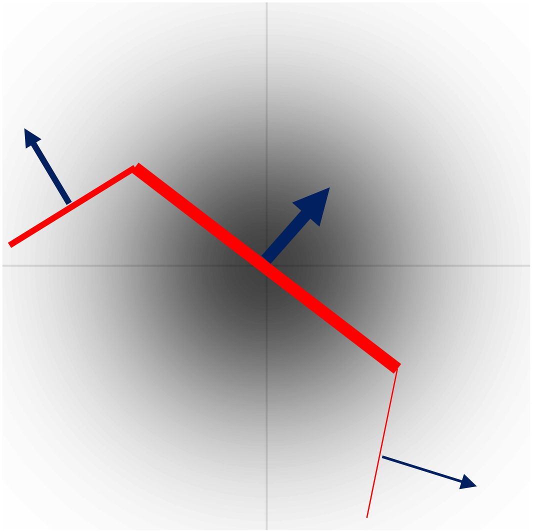




[Shiu 83]:
algorithm for obtaining **integers** *a* and *b* that approximate cosθ and sinθ

Using Rotation Matrices

- How do we **choose a good rotation?**



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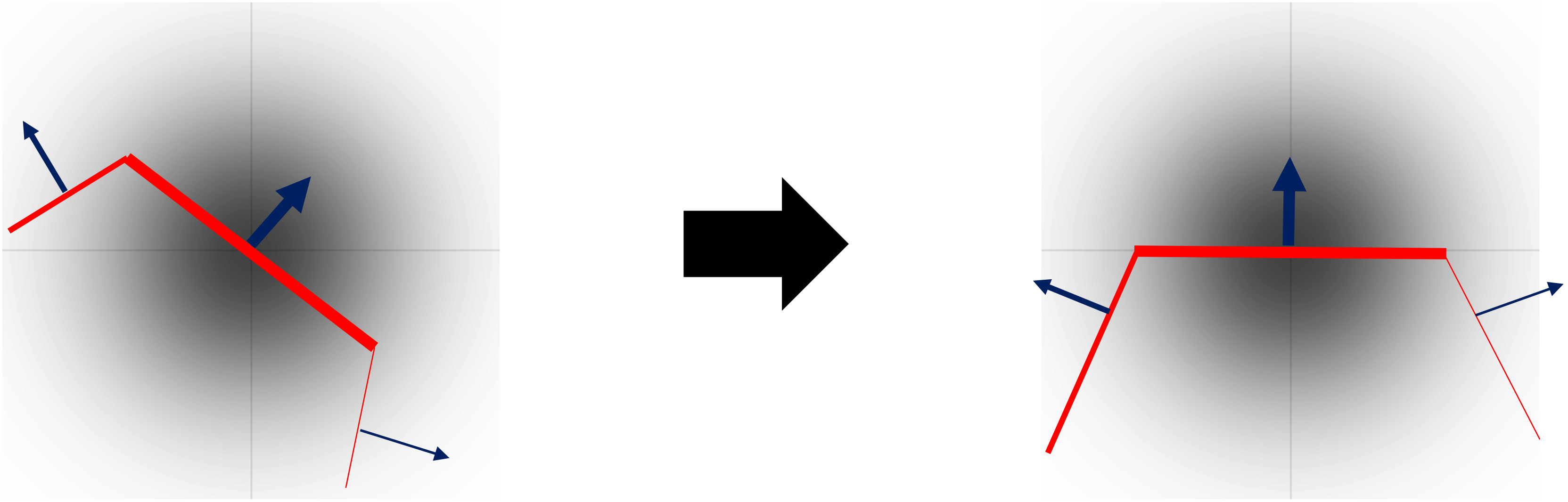

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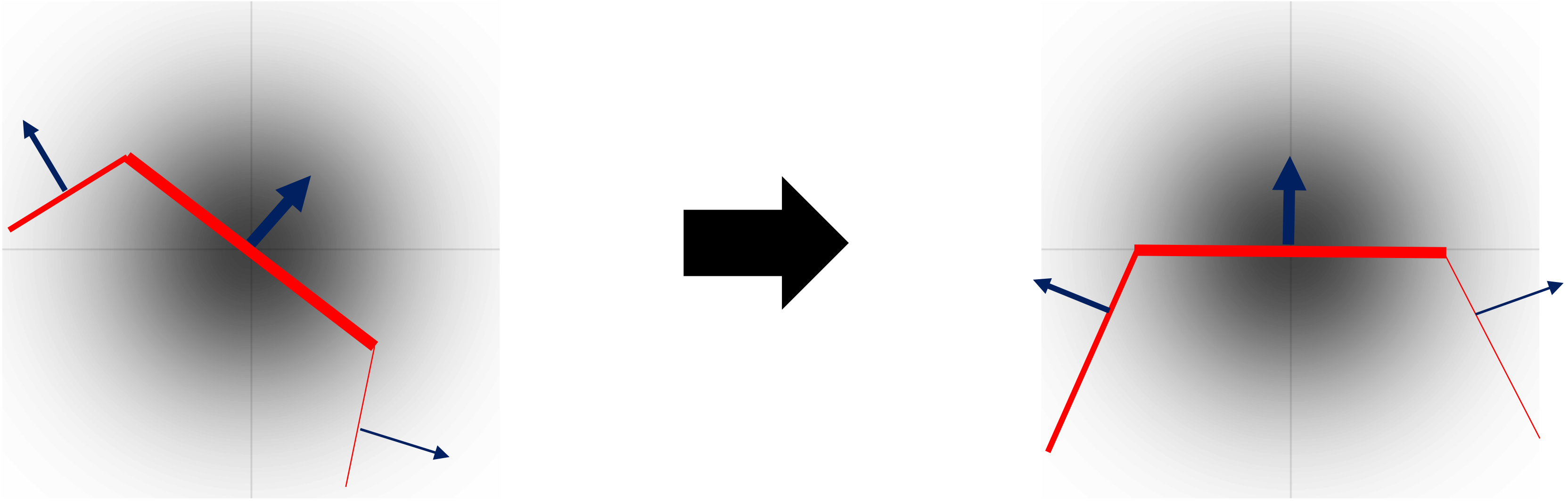
  - Compose from rational Givens rotations



Implementation

- Python
- Z3

Evaluation

- Randomly generated formulas
  - Generated 8 kinds of formula—40 instances each.
  - Sampled rectangles 100s for each instance.
  - Average improvement between **17%** (3-variable formulas) and **1500%** (7-variable formulas)

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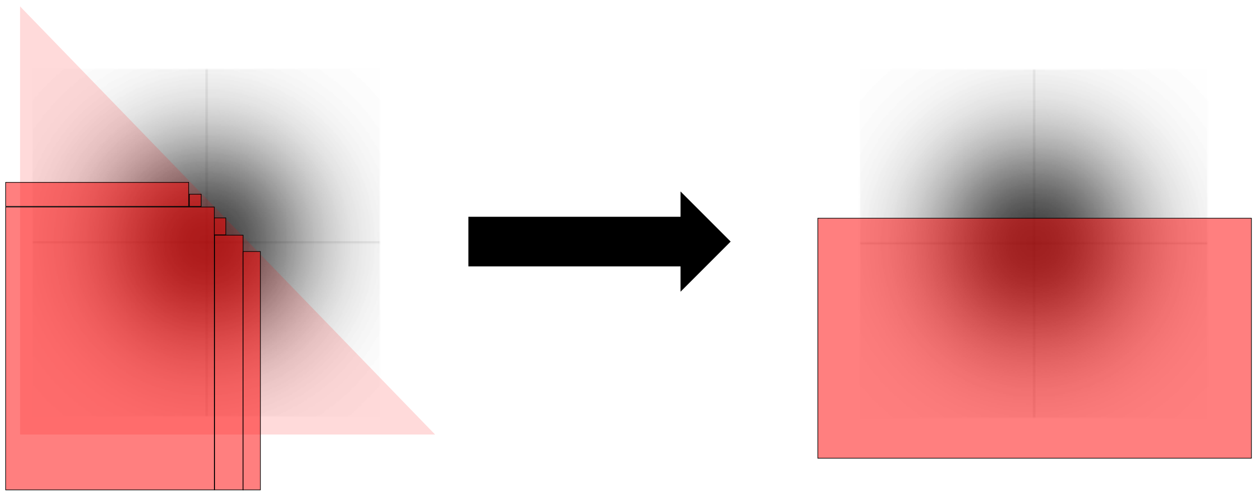
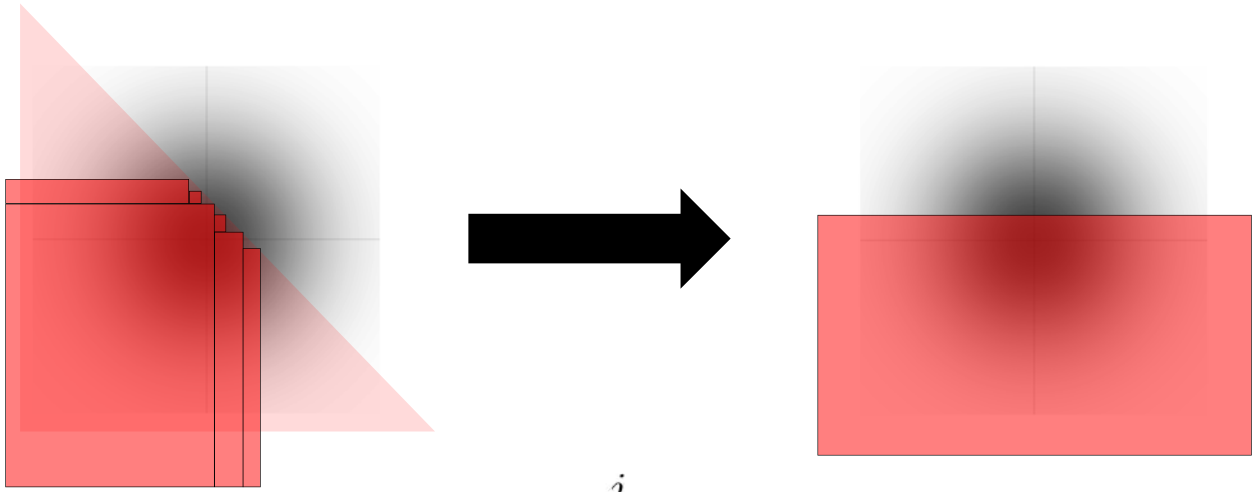
- Probabilistic programs from literature
  - Improved efficiency for 10 of 12 example programs.





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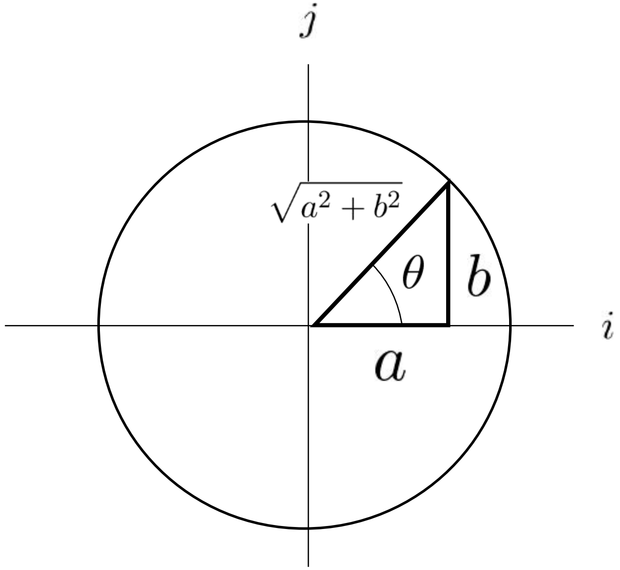
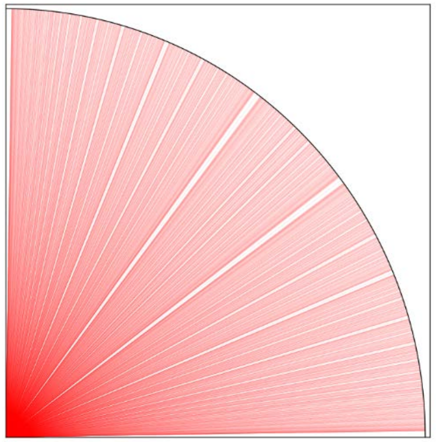
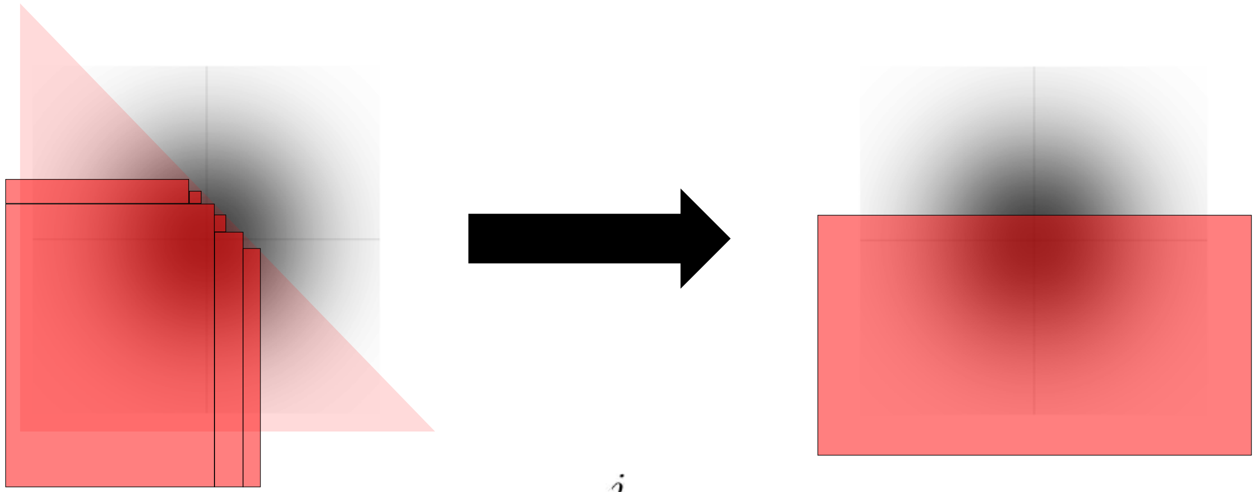




Questions?



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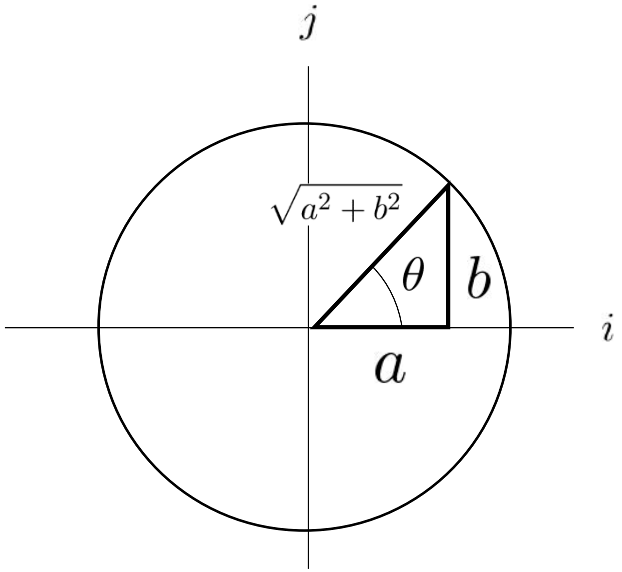
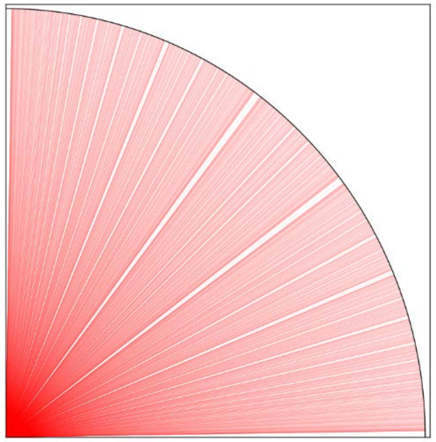
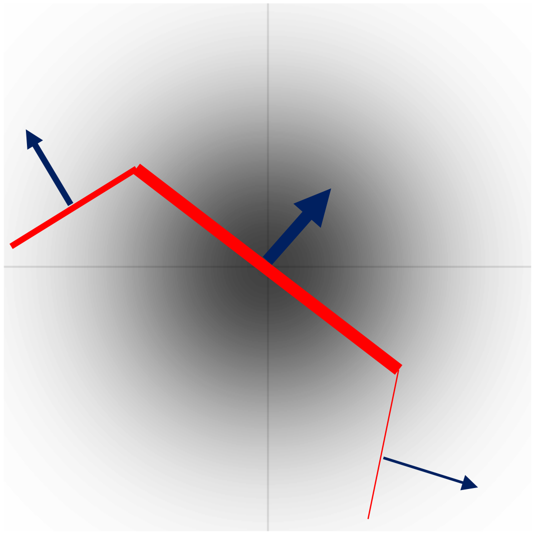




Extras

Using Rotation Matrices

- How do we **choose a good rotation?**



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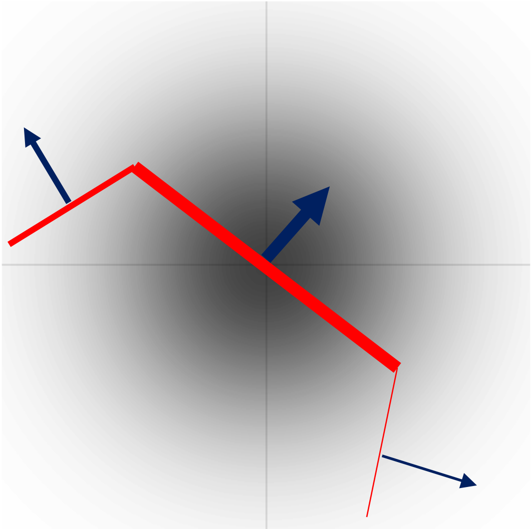

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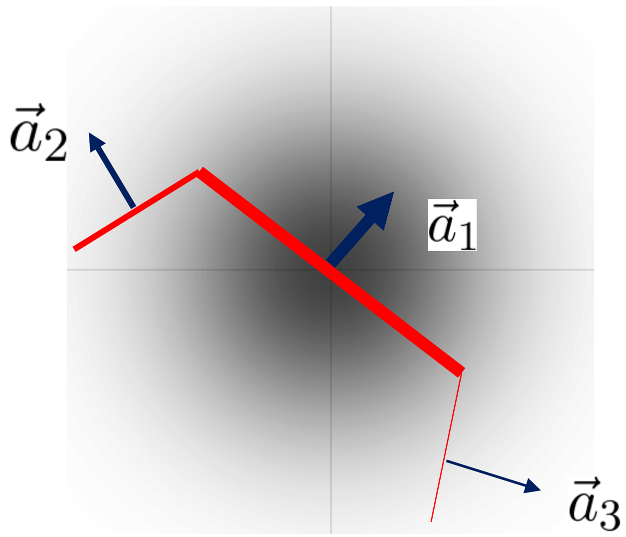

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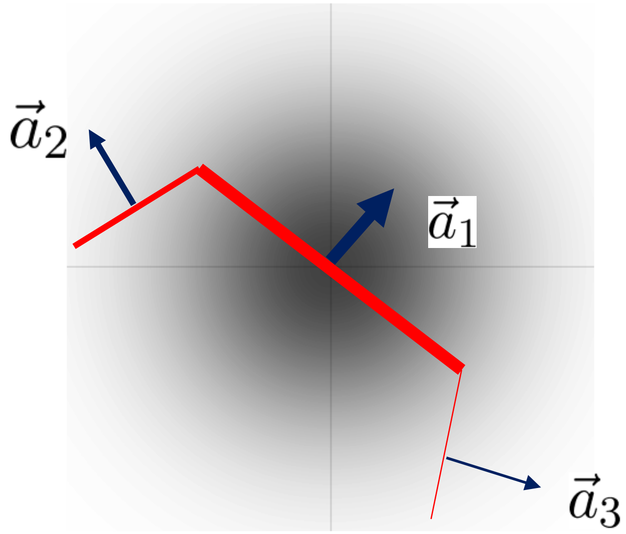

$$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$$

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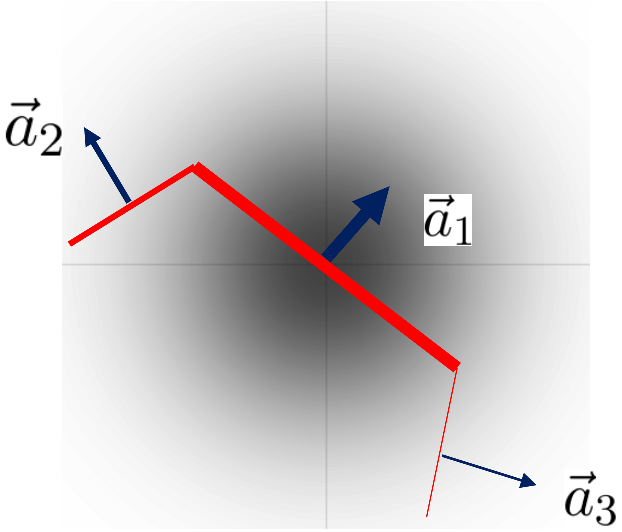

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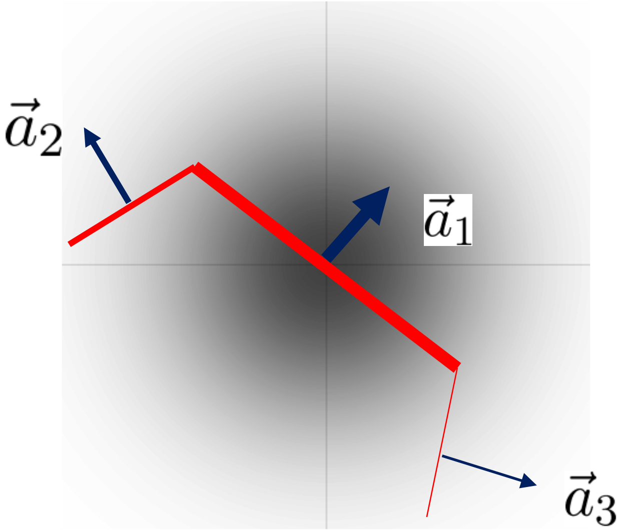

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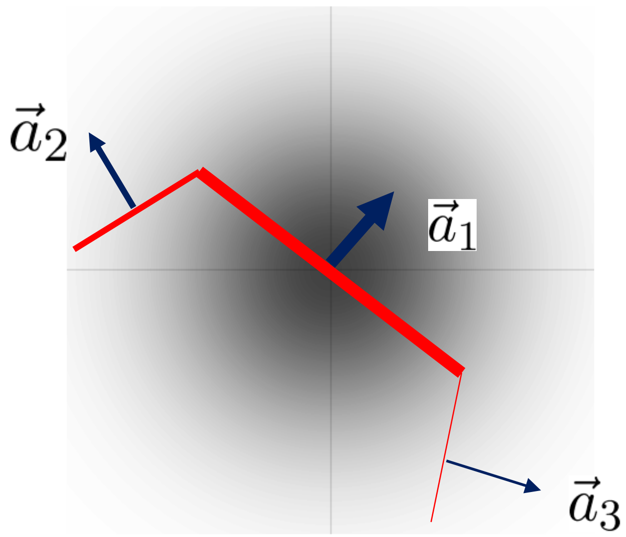

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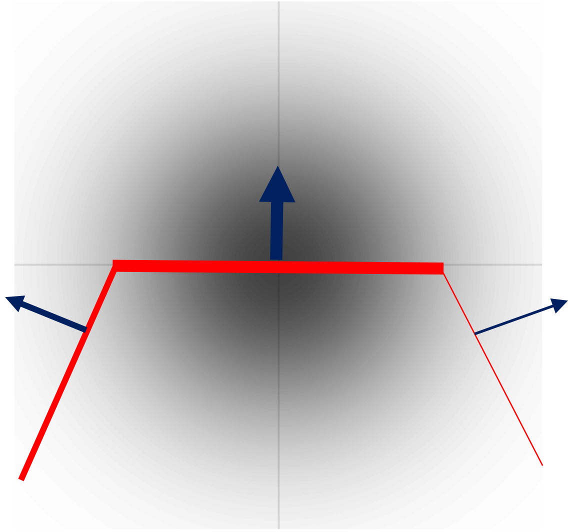

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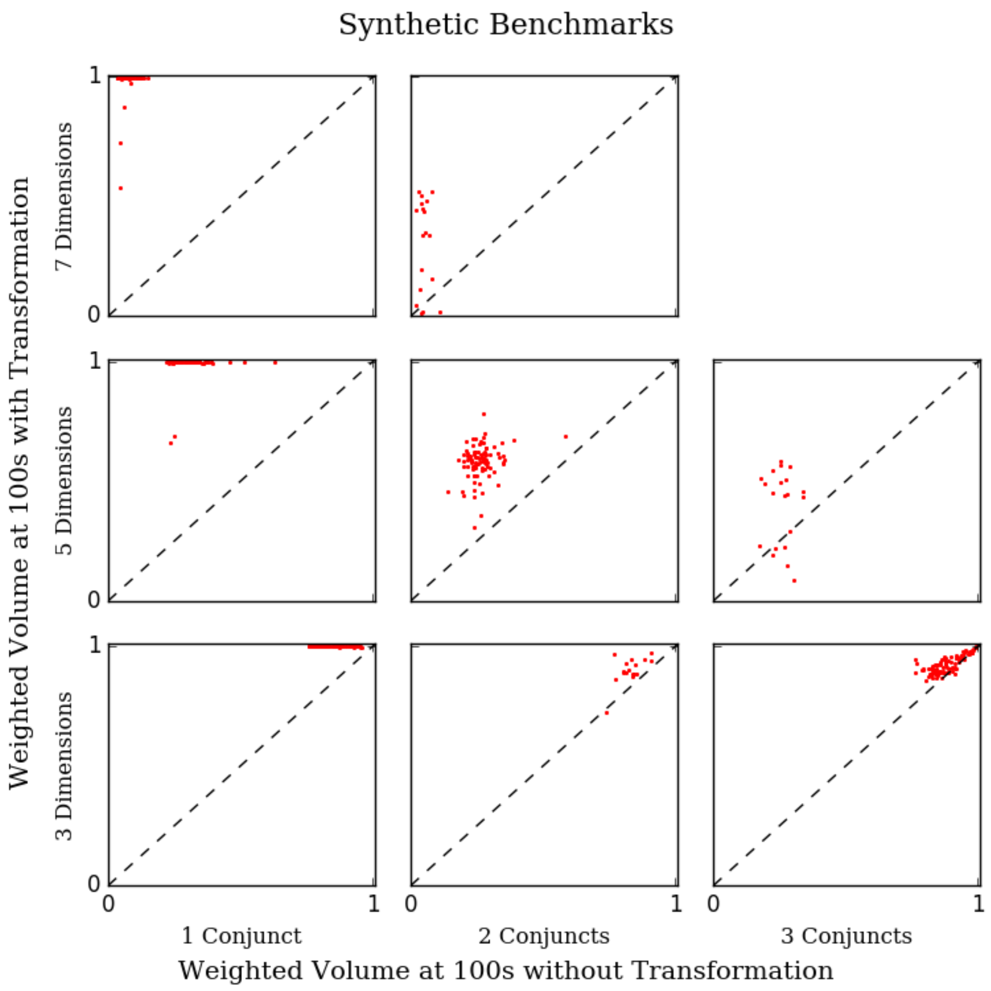

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- Randomly generated formulas
- Probabilistic programs

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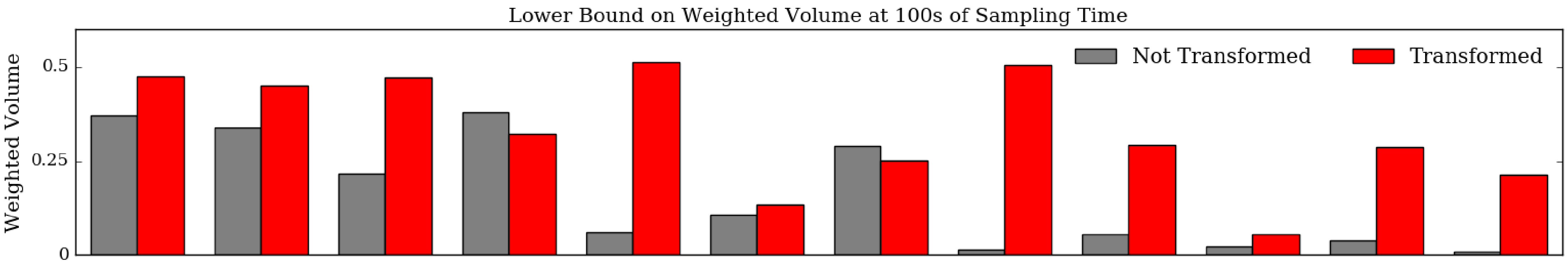

Benchmark	1d-2s-simp	1d-2s-b2	1d-3s-simp	1d-3s-b2	2d-2s-simp	2d-2s-b	2d-2s-b2	2d-3s-simp	3d-2s-simp	3d-2s-b	3d-2s-b2	3d-3s-simp
# Predicates	14	36	18	50	25	61	55	31	36	90	76	45
# Free Vars	3	5	4	7	6	10	8	8	9	15	11	8
# Quant. Vars	12	27	16	36	23	46	43	29	34	67	61	43

Conclusions

- An **orthogonal transformation preprocessing step** can improve sampling efficiency for Weighted Model Integration methods.

- We addressed challenges and limitations of this approach.

Future Work

- Can we **choose better rotations** for this preprocessing step?

- Are there new ways to perform exact Weighted Model Integration, **beyond rectangular decomposition**? E.g., numerical quadrature?